Exam Prep!

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation.

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}$

so $T\begin{pmatrix}3\\4\end{pmatrix} = 3\begin{pmatrix}5\\6\end{pmatrix} + 4\begin{pmatrix}0\\3\end{pmatrix}$

To find the matrix use: $\left[T(\vec{e}_1) \cdots T(\vec{e}_n)\right]$

Notes

Def - A **basis** for a subspace $W \subset \mathbb{R}^n$ is a set $\beta = \{\vec{v}_1, \dots, \vec{v}_k\} \subset W$ s.t.

(i) β is linearly independent

(ii) $W = \operatorname{span} \beta$

Example

Suppose $A = [\vec{a}_1 \ \dots \ \vec{a}_6]$ is a 3×6 matrix row equivalent to

$$\begin{bmatrix} 0 & 1 & -2 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Give Bases for image + kernal of A

$A\vec{x} = \vec{0}$

$$\begin{cases} x_2 = 2x_3 - x_4 - 4x_6 \\ x_5 = -7x_6 \end{cases}$$

+ all other variables are free, so $\begin{pmatrix}x_1\\ \vdots \\ x_6\end{pmatrix} = x_1\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix} + x_3\begin{pmatrix}0\\2\\1\\0\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\-1\\0\\1\\0\\0\end{pmatrix} + x_6\begin{pmatrix}0\\-4\\0\\0\\-7\\1\end{pmatrix}$

So basis for Ker A is

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-4\\0\\0\\-7\\1\end{pmatrix}$$

These have no non-trivial linear relations because each vector has 1 in some coordinate (coordinate to a free variable) but 0's in coords coordinates corresponding to other free variables.

Image of A: all vecs of form $A\vec{x}$ or $x_1\vec{a}_1 + \dots + x_6\vec{a}_6$

i.e. span $(\vec{a}_1 \dots \vec{a}_6)$

But $\vec{a}_1, \dots, \vec{a}_6$ are <u>not</u> linearly independent

e.g. $\vec{x} = \begin{pmatrix} 0 \\ -4 \\ 0 \\ 0 \\ -7 \\ 1 \end{pmatrix}$ satisfies $A\vec{x} = \vec{0}$

$-4\vec{a}_2 - 7\vec{a}_5 + \vec{a}_6 = \vec{0}$

$\vec{a}_6 = 4\vec{a}_2 + 7\vec{a}_5$

In fact, you can use the same trick to write any col $\vec{a}$, corresponding to a free variable as a combination of columns corresponding to pivot variables.

<u>Upshot</u> $\to$ $\underbrace{\vec{a}_2, \vec{a}_5}$ is a basis for the image of A.

these columns are the columns of A that correspond to pivot columns of A in RREF.

Thm (Fundamental Theorem of <u>Linear</u> Algebra)

(i) Every non-trivial subspace of $\mathbb{R}^n$ has a basis

(ii) Any two bases for the same subspace have the same # of vectors.

<u>Def</u> - The <u>dimension</u> of a non-trivial subspace $W \subset \mathbb{R}^n$ is the # of vecs in a basis for W.

<u>Lemma</u> - If $W = \text{span}(\vec{v}_1, \dots, \vec{v}_k) \subset \mathbb{R}^n$ and $\vec{w}_1, \dots, \vec{w}_l \in W$ are linearly independent then $l \leq k$.

$\boxed{\text{e.g.}}$

$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ is a basis for $\mathbb{R}^3$.

$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\11\\56\end{pmatrix}, \begin{pmatrix}\pi\\e\pi\\\sqrt{2}\end{pmatrix}$ is another basis for $\mathbb{R}^3$.

but $\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\3\\7\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ are not basis.